A 4/3-approximation for TSP on cubic 3-edge-connected graphs

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1 Introduction

We consider the travelling salesman problem on metrics which can be viewed as the shortest path metric of an undirected graph with unit edge-lengths. Finding a TSP tour in such a metric is then equivalent to finding a connected Eulerian subgraph in the underlying graph. Since the length of the tour is the number of edges in this Eulerian subgraph our problem can equivalently be stated as follows: Given an undirected, unweighted graph $G = (V, E)$ find a connected Eulerian subgraph, $H = (V, E')$ with the fewest edges. Note that H could be a multigraph.

In this paper we consider the special case of the problem when G is 3-regular (also called cubic) and 3-edge-connected. Note that the smallest Eulerian subgraph contains at least $n = |V|$ edges. In fact, in the shortest path metric arising out of such a graph the Held-Karp bound for the length of the TSP tour would also be n. This is because we can obtain a fractional solution to the sub-tour elimination LP (which is equivalent to the Held-Karp bound) of value n by assigning 2/3 to every edge in G.

Improving the approximation ratio for metric-TSP beyond 3/2 is a long standing open problem. For the metric completion of cubic 3-edge connected graphs Gamarnik et.al. [1] obtained an algorithm with an approximation guarantee slightly better than 3/2. The main result of this paper is to improve this approximation guarantee to 4/3 by giving a polynomial time algorithm to find a connected Eulerian subgraph with at most 4n/3 edges. This matches the conjectured integrality gap for the sub-tour elimination LP for the special case of these metrics.

2 Preliminaries

Let n be the number of vertices of the given graph G. Let $d(x)$ denote the degree of x. A 2-factor in G is a subset of edges X such that every vertex has degree 2 in X. Let σ(X) denote the minimum size of components of X. Given two distinct edges $e_1 = x_1v$ and $e_2 = x_2v$ incident on a vertex v, let ${G_v}^{e_1,e_2}$ denote the graph obtained by replacing e_1, e_2 by the edge x_1x_2. The vertex v is said to be *split off*. We call a cut $(S, \overline{S})$ *essential* when both S and $\overline{S}$ contain at least one edge each.

We will need the following results for our discussion

Lemma 1 (Peterson[4]). *Every bridgeless cubic graph has a 2-factor.*

Lemma 2 (Mader[3]). *Let $G = (V, E)$ be a k-edge-connected graph, $v \in V$ with $d(v) \geq k + 2$. Then there exists edges $e_1, e_2 \in E$ such that $G_v^{e_1,e_2}$ is homeomorphic to a k-edge-connected graph.*

Lemma 3 (Jackson, Yoshimoto[2]). *Let G be a 3-edge-connected graph with n vertices. Then G has a spanning even subgraph in which each component has at least $\min(n, 5)$ vertices.*

3 Algorithm

Our algorithm can be broadly split into three parts. We first find a 2-factor of the cubic graph that has no 3-cycles and 4-cycles. Next, we compress the 5-cycles into 'super-vertices' and split them using Lemma 2 to get a cubic 3-edge-connected graph G$'$ again. Repeatedly applying the first part on G' and compressing the five cycles gives a 2-factor with no 5-cycle on the vertices of the original graph. We 'expand' back the super-vertices to form X that is a subgraph of G. We finally argue that X can be modified to get a connected spanning even multi-graph using at most 4/3(n) edges.

The starting point of our algorithm is Theorem 3 [2]. In fact [2] proves the following stronger theorem.

Theorem 1. *Let G be a 3-edge-connected graph with n vertices, u_2 be a vertex of G with $d(u_2) = 3$, and $e_1 = u_1u_2; e_2 = u_2u_3$ be edges of G. (it may be the case that $u_1 = u_3$). Then G has a spanning even subgraph X with $\{e_1, e_2\} \subset E(X)$ and $\sigma(X) \geq \min(n, 5)$.*

The proof of this theorem is non-constructive. We refer to the edges e_1, e_2 in the statement of the theorem as "required edges". We now discuss the changes required in the proof given in [2] to obtain a polynomial time algorithm which gives the subgraph X with the properties as specified in Theorem 1. Note that we will be working with a 3-regular graph (as against an arbitrary graph of min degree 3 in [2]) and hence the even subgraph X we obtain will be a 2-factor.

1. If G contains a non-essential 3-edge cut then we proceed as in the proof of Claim 2 in [2]. This involves splitting G into 2 graphs G_1, G_2 and suitably defining the required edges for these 2 instances so that the even subgraphs computed in these 2 graphs can be combined. This step is to be performed whenever the graph under consideration has an essential 3-edge cut.

2. Since G is 3-regular we do not require the argument of Claim 6.

3. Since G has no essential 3-edge cut and is 3-regular, a 3-cycle in G implies that G is K_4. In this case we can find a spanning even subgraph containing any 2 required edges.

4. The process of eliminating 4-cycles in the graph involves a sequence of graph transformations. The transformations are as specified in [2] but the order in which the 4-cycles are considered depends on the number of required edges in the cycle. We first consider all such cycles which do not have any required edges, then cycles with 2 required edges and finally cycles which have one required edge.

   Since with each transformation the number of edges and vertices in the graph reduces we would eventually terminate with a graph, say G', with girth 5. We find a 2-factor in G', say X' and undo the transformations (as specified in [2]) in the reverse order in which they were done to obtain a 2-factor X in the original graph G which has the properties of Theorem 1.

Suppose the 2-factor obtained X contains a 5-cycle C. We compress the vertices of C into a single vertex, say v_C, and remove self loops. v_C has degree 5 and we call this vertex a *super-vertex*. We now use Lemma 2 to replace two edges x_1v_C and x_2v_C incident at v_C with the edge x_1x_2 while preserving 3-edge connectivity. The edge x_1x_2 is called a *super-edge*. Since the graph obtained is cubic and 3-edge connected we can once again find a 2-factor, each of whose cycles has length at least 5. If there is a 5-cycle which does not contain any super-vertex or super-edge we compress it and repeat the above process. We continue doing this till we obtain a 2-factor, say X, each of whose cycles is either of length at least 6 or contains a super-vertex or a super-edge.

In the 2-factor X we replace every super-edge with the corresponding edges. For instance the super-edge x_1x_2 would get replaced by edges x_1v_C and x_2v_C where v_C is a super-vertex obtained by collapsing the vertices of a cycle C. After this process X is no more a 2-factor but an even subgraph. However, the only vertices which have degree more than 2 are the super-vertices and they can have a maximum degree 4. Let X denote this even subgraph.

Consider some connected component W of X. We will show how to expand the super-vertices in W into 5-cycles to form an Eulerian subgraph with at most $\lfloor 4|W'|/3 \rfloor - 2$ edges, where $|W'|$ is number of vertices in the expanded component. For each component we will use 2 more edges to connect this component to the other components to obtain a connected Eulerian subgraph with at most $\lfloor 4n/3 \rfloor - 2$ edges. Note that the subgraph we obtain may use an edge of the original graph at most twice.

We now consider two cases depending on whether W contains a super-vertex.

1. W has no super-vertices. Then, W is a cycle with at least 6 vertices and hence Eulerian. Since $|W|/3 \geq 2$ the claim follows.

2. W has at least one super-vertex, say s. We will discuss the transformations for a single super-vertex and this will be repeated for the other super-vertices. Note that s has degree 2 or 4.

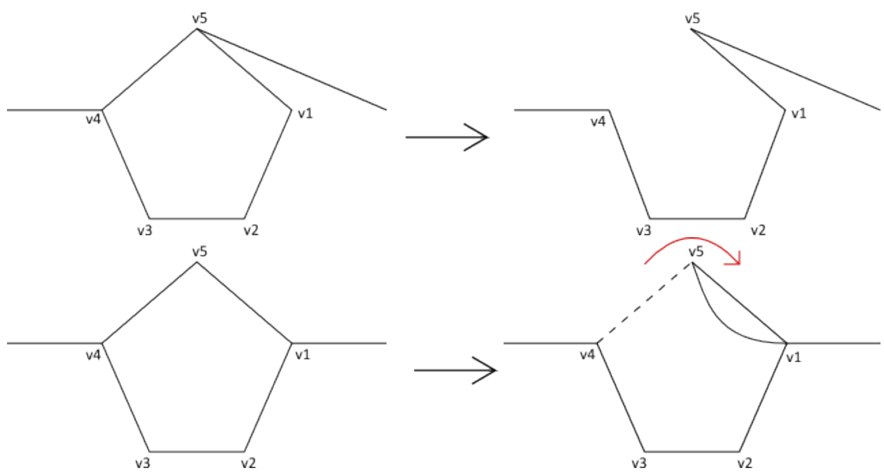

Figure 1: On Expanding a super-vertex with degree 2

If s has degree 2, then the 2 edges incident on the 5-cycle corresponding to s would be as in Figure 1. In both cases we obtain an Eulerian subgraph. By this transformation we have added 4 vertices and at most 5 edges to the subgraph W.

Suppose the super-vertex s has degree 4 in the component W. W may not necessarily be a component of the subgraph X as it might have been obtained after expanding a few super-vertices, but that will not effect our argument. Let C be the 5-cycle corresponding to this super-vertex and let v_1, v_2, v_3, v_4, v_5 be the vertices on C (in order). Further let v'_i be the vertex not in C adjacent to v_i. Let $v_5v'_5$ be the edge incident on C that is not in the subgraph W.

We replace the vertex s in W with the cycle C and let W' be the resulting subgraph. Note that by dropping edges v_1v_2 and v_3v_4 from W' we obtain an Eulerian subgraph which includes all vertices of C. However, this subgraph may not be connected as it could be the case that edges v_1v_2 and v_3v_4

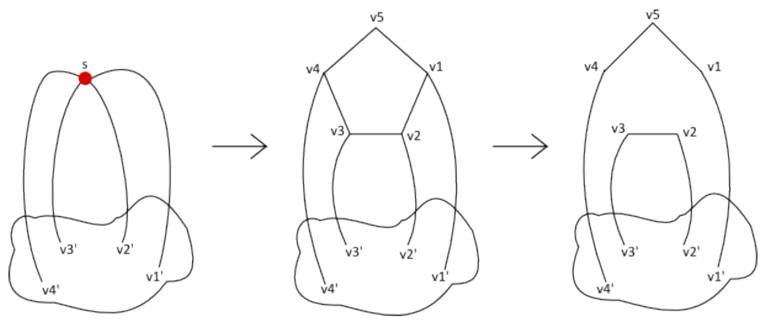

Figure 2: Expanding a super-vertex with degree 4 when v_1v_2 and v_3v_4 do not form a 2-edge-cut of the sub-graph constructed till now.

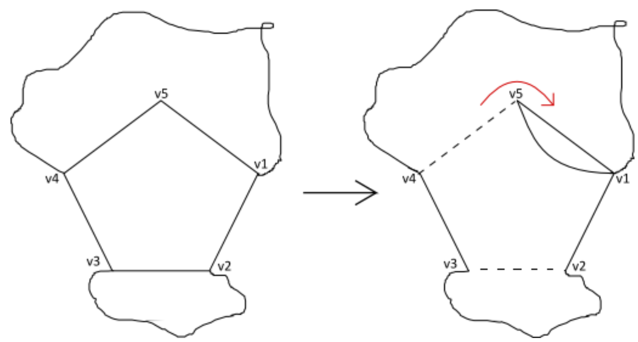

Figure 3: Expanding a super-vertex with degree 4 when v_1v_2 and v_3v_4 form a 2-edge-cut.

form an edge-cut in W'. If this is the case then we apply the transformation as shown in Figure 3. This ensures that W' remains connected and is Eulerian. Note that as a result of this step we have added 4 vertices and at most 4 edges to the subgraph W.

Let W' be the component obtained by expanding all the super-vertices in W. Suppose initially, component W had k_1 super-vertices of degree 2, k_2 super-vertices of degree 4 and k_3 vertices of degree 2. This implies W had k_1 + $2k_2$ + k_3 edges. On expanding a super-vertex of degree 2, we add 5 edges in the worst case. On expanding a super-vertex of degree 4, we add 4 edges in the worst case. So, the total number of edges in W' is at most $6k_1 + 6k_2 + k_3$ while the number of vertices in W' is exactly $5k_1 + 5k_2 + k_3$. Note that $k_1 + k_2 + k_3 \geq 5$ and if $k_1 + k_2 + k_3 = 5$ then $k_1 + k_2 \geq 1$. Hence, $2k_1 + 2k_2 + k_3 \geq 6$ and this implies that the number of edges in W' is at most $\lfloor 4|V(W')|/3 \rfloor - 2$.

4 Conclusions

We show that any cubic 3-edge connected graph contains a connected Eulerian subgraph with at most 4n/3 edges. It is tempting to conjecture the same for non-cubic graphs especially since the result in [2] holds for all 3-edge connected graphs. The example of a $K_{3,n}$ demonstrates that this conjecture would be false. A $K_{3,n}$ is 3-edge connected and any connected Eulerian subgraph contains at least $2n$ edges.

References

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